Math 279 Lecture 27 Notes

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1 Algebraic Structure in Our Regularity Structure

1.1 Products structures in rough path theory

What kind of algebra leads to our group G and the form $\Gamma_{x,y}$? We first discuss Hopf algebras.[1] In fact, Connes and Kreimer have observed that a suitable Hopf algebra on the polynomials of "decorated trees" can be used to explain renormalization phenomena in quantum field theory and Feynman diagrams.

To motivate the role of such algebras, let us go back to our rough path theory first. Indeed, if we have a path $x = (x^1, \dots, x^\ell) : [0, T] \to \mathbb{R}^\ell$, then we need a candidate for

$$\langle \mathbf{x}(s,t), e_{i_1 \cdots i_r} \rangle = \int_s^t \int_s^{s_1} \int_s^{s_r} dx^{i_1}(x_1) \cdots dx^{i_r}(s_r) =: a_{i_1,\dots,i_r}.$$

What we have in mind is that we choose a lift for the path x that is tensor-valued, and this condition yields the $e_{i_1 \cdots i_r}$ component of such a tensor. More precisely, if the space of tensors $T(\mathbb{R}^d) = \mathbb{R} \oplus \ell^\ell \oplus \cdots \oplus (\mathbb{R}^\ell)^{\otimes n} \oplus \cdots$, then

$$\overline{x}(s,t) = \sum_{i_1,\dots,i_r} a_{i_1,\dots,i_r} e_{i_1,\dots,i_k},$$

where $e_{i_1,\dots,i_r} = e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_r}$ and the $e_1 \cdot e_\ell$s form a basis for $\mathbb{R}^\ell$. As we have seen before, if $x \in \mathcal{C}^\alpha$ with $\frac{1}{n} < \alpha \leq \frac{1}{n-1}$, then we can truncate our tensor at level $n-1$. Note that if $x \in \mathcal{C}^\alpha$, then the type of regularity we have is

$$|\langle \mathbf{x}(s,t), e_{i_1 \cdots i_r} \rangle| \lesssim |s-t|^{r\alpha}.$$

Recall that Chen's relation becomes $\mathbf{x}(s,u) \otimes \mathbf{x}(u,t) = \mathbf{x}(s,t)$, which allows us to only consider $\mathbf{x}(0,t)$ because $\mathbf{x}(s,t) = \mathbf{x}(0,s)^{-1} \otimes \mathbf{x}(0,t)$.

[1]Historically, Hopf was studying homology and cohomology on Lie groups, where the additional multiplication of the Lie group gave extra algebraic structure to the homology.

Recall that when $\frac{1}{3} \le \alpha < \frac{1}{2}$, for a metric path, we have

$$\mathbb{X}(s,t) + \mathbb{X}^*(s,t) = \mathbf{x}(s,t) \otimes \mathbf{x}(s,t),$$

or

$$\langle \mathbf{x}(s,t), e_{i,j} + e_{j,i} \rangle = \langle \mathbf{x}(s,t), e_i \rangle \langle \mathbf{x}(s,t), e_j \rangle.$$

However, for low α, the geometric condition becomes

$$\langle \mathbf{x}(s,t), e_{i_1,\dots,i_r} \rangle \langle \mathbf{x}(s,t), e_{j_1 \cdots j_\ell} \rangle = \langle \mathbf{x}(s,t), e_{i_1,\dots,i_r} \sqcup\!\sqcup e_{j_1,\cdots,j_\ell} \rangle,$$

where $a \sqcup\!\sqcup b$ means the **shuffle product** of a and b:

$$e_{i_1 \cdots i_r} \sqcup\!\sqcup e_{j_1 \cdots j_\ell} = \sum e_{k_1 \cdots k_{r+\ell}},$$

where $k_1, \cdots, k_{r+\ell}$ is obtained from $i_1, \dots, i_r, j_1, \dots, j_\ell$ by interleaving them without changing the original order. So there are exactly $\frac{(k+\ell)!}{k!\ell!}$ many terms.

Example 1.1.

$$e_i \sqcup\!\sqcup e_j = e_{i,j} + e_{j,i}$$

Example 1.2.

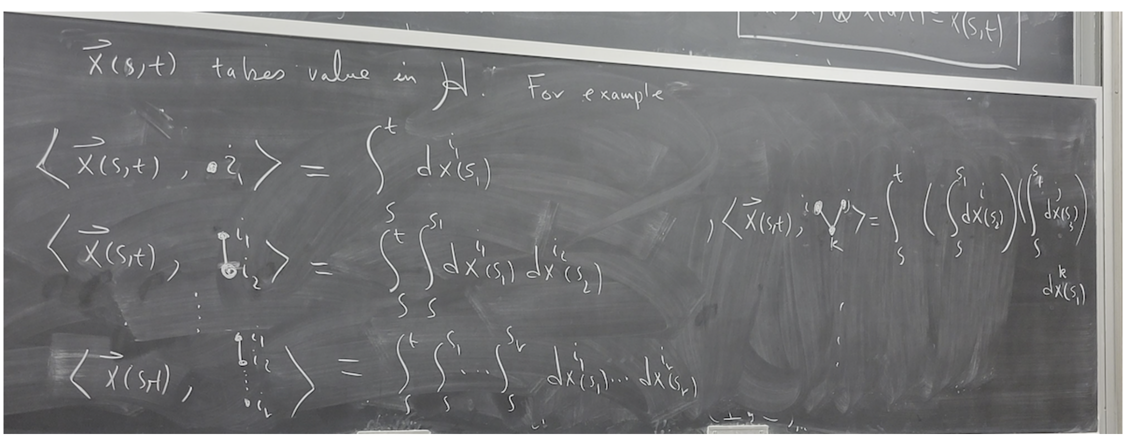
$$e_i \sqcup\!\sqcup e_{j,k} = e_{i,j,k} + e_{j,i,k} + e_{j,k,i}$$

In summary, for a geometric path, we have two products on $T(\mathbb{R}^\ell)$:

$$\begin{cases} \mathbf{x}(s,u) \otimes \mathbf{x}(u,t) = \mathbf{x}(s,t) \\ \langle \mathbf{x}(s,t), a \rangle \langle \mathbf{x}(s,t), b \rangle = \langle \mathbf{x}(s,t), a \sqcup\!\sqcup b \rangle. \end{cases}$$

How about for a nongeometric path? This has been worked out by Gubinelli and involves Connes-Kreimer's Hopf algebra (the theory of branched paths). In this case, the right space is not the tensor algebra, rather the algebra of polynomials of decorated binary trees with decorations/labels selected from $\{1, \dots, \ell\}$. Write $\mathcal{H}$ for this space. Now $\underline{\mathrm{x}}(s,t)$ takes values in $\mathcal{H}$.

Example 1.3.



What happens to Chen's relation? There is an other product, the convolution product $\star$ that would allow us to represent Chen's relation as

$$\mathbf{x}(s,u) \star \mathbf{x}(u,t) = \mathbf{x}(s,t).$$

Before we define this, let us discuss the notion of Hopf algebras first.

1.2 Hopf algebras

Let $\mathcal{H}$ be an algebra with unit $\mathbf{1}$ and product $\cdot$. Also suppose we have an algebra on $\mathcal{H}^*$. Let us write $\langle \cdot, \cdot \rangle : \mathcal{H}^* \times \mathcal{H} \to \mathbb{R}$ for the pairing between $\mathcal{H}$ and $\mathcal{H}^*$. We write $f \star g$ for the product on $\mathcal{H}^*$ and $\mathbf{1}^*$ for its unit. We use pairing to turn $\star$ into a "coproduct" on $\mathcal{H}$. Also, inversion in $\mathcal{H}^*$ can be translated into a suitable notion on $\mathcal{H}$. When this is done successfully, we have a **Hopf algebra**.

Here is the idea: if $f, g \in \mathcal{H}^*$ and $h \in \mathcal{H}$, then

$$\underbrace{\langle f \star g, h \rangle}_{\text{pairing of } \mathcal{H}^*, \mathcal{H}} = \underbrace{\langle f \otimes g, C(h) \rangle}_{\text{pairing of } \mathcal{H}^* \otimes \mathcal{H}^*, \mathcal{H} \otimes \mathcal{H}} .$$

If we find such a C, then we have a bialgebra. Here, C is our coproduct $C : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$.

We now try the idea of the inverse: If $f \in \mathcal{H}^*$ is invertible, then $f \star f^{-1} = \mathbf{1}^*$. We wish to find an operator $\mathcal{S} : \mathcal{H} \to \mathcal{H}$ so that $\mathcal{S}^* : \mathcal{H}^* \to \mathcal{H}^*$ is exactly $\mathcal{S}^*(f) = f^{-1}$. If such an operator exists, then we should have

$$\begin{aligned}
\langle \mathbf{1}^*, h \rangle &= \langle f \star \mathcal{S}^* f, h \rangle \\
&= \langle f \otimes \mathcal{S}^* f, C(h) \rangle \\
&= \langle ((\mathrm{id} \otimes \mathcal{S}^*)(f \otimes f), C(h) \rangle \\
&= \langle f \otimes f, (\mathrm{id} \otimes \mathcal{S}) C(h) \rangle.
\end{aligned}$$

Assume $\langle f, \mathbf{1} \rangle = 1$. Then if we require $(\mathrm{id} \otimes \mathcal{S}) C(h) = \langle \mathbf{1}^*, h \rangle \mathbf{1}$, then we have our $\mathcal{S}$. And if such a C and $\mathcal{S}$ exist, we have a Hopf algebra.

Definition 1.1. A **Hopf algebra** is an algebra $\mathcal{H}$ with a coproduct $C : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$ and an operator $\mathcal{S} : \mathcal{H} \to \mathcal{H}$ such that

$$(\mathrm{id} \otimes \mathcal{S}) C(h) = \langle \mathbf{1}^*, h \rangle \mathbf{1}.$$

Example 1.4. Let $\mathcal{H}$ be the algebra generated from $(\partial_i = \frac{\partial}{\partial x_i} : i = 1, \dots, d)$ with the product given by the composition: $D = \partial_{i_1, \dots, i_k} = \partial_{i_1} \cdots \partial_{i_k}$. $\mathcal{H}^*$ is the space of smooth functions with pointwise multiplication. The pairing is $\langle f, D \rangle = (Df)(0)$.

We claim that this is a Hopf algebra. One can show by integration by parts that

$$C(\partial_i) = \mathrm{id} \otimes \partial_i + \partial_i \otimes \mathrm{id},$$

$$C(\partial_i \partial_j) = \mathrm{id} \otimes \partial_{i,j} + \partial_i \otimes \partial_j + \partial_j \otimes \partial_i + \partial_{i,j} \otimes \mathrm{id}.$$